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Complexified von Roos Hamiltonian's η-weak-pseudo-Hermiticity, isospectrality and exact solvability

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Abstract

A complexified von Roos Hamiltonian is considered and a Hermitian first-order intertwining differential operator is used to obtain the related position-dependent mass η-weak-pseudo-Hermitian Hamiltonians. Using a Liouvillean-type change of variables, the η-weak-pseudo-Hermitian von Roos Hamiltonians H_x are mapped into the traditional Schrödinger Hamiltonian form H_q, where exact isospectral correspondence between H_x and H_q is obtained. Under a '*user-friendly*' position-dependent-mass setting, it is observed that for each exactly solvable η-weak-pseudo-Hermitian *reference*-Hamiltonian H_q there is a set of exactly solvable η-weak-pseudo-Hermitian *isospectral target*-Hamiltonians H_x. A non-Hermitian $\mathcal{PT}$-symmetric Scarf II and a non-Hermitian periodic-type $\mathcal{PT}$-symmetric Samsonov–Roy potentials are used as *reference* models and the corresponding η-weak-pseudo-Hermitian *isospectral target*-Hamiltonians are obtained.

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1. Introduction

Subjected to von Roos constraint $\alpha+\beta+\gamma=-1$; $\alpha,\beta,\gamma\in\mathbb{R}$, the von Roos position-dependent-mass (PDM) Hamiltonian [1–12] reads

$$H=-\partial_x\left(\frac{1}{M(x)}\right)\partial_x+\tilde{V}(x), \qquad (1)$$

with

$$\tilde{V}(x)=\frac{1}{2}(1+\beta)\frac{M''(x)}{M(x)^2}-[\alpha(\alpha+\beta+1)+\beta+1]\frac{M'(x)^2}{M(x)^3}+V(x), \qquad (2)$$

and primes denote derivatives. An obvious profile change of the potential $\tilde{V}(x)$ obtains as α, β and γ change, manifesting in effect an ordering ambiguity conflict in the process of choosing a unique kinetic energy operator

$$T = -\tfrac{1}{2}[M(x)^{\alpha}\partial_x M(x)^{\beta}\partial_x M(x)^{\gamma} + M(x)^{\gamma}\partial_x M(x)^{\beta}\partial_x M(x)^{\alpha}] \tag{3}$$

Hence, α, β and γ are usually called the von Roos ambiguity parameters. Yet, such PDM-quantum-particles (i.e. $M(x) = m_{\circ}m(x)$) are used in the energy density many-body problem, in the determination of the electronic properties of semiconductors and quantum dots [1–5].

Regardless of the continuity requirements on the wavefunction at the boundaries of abrupt herterojunctions between two crystals [6] and/or Dutra's and Almeida's [7] reliability test, there exist several suggestions for the kinetic energy operator in (3). We may mention the Gora's and Williams' ($\beta = \gamma = 0, \alpha = -1$) [8], Ben Daniel's and Duke's ($\alpha = \gamma = 0, \beta = -1$) [9], Zhu's and Kroemer's ($\alpha = \gamma = -1/2, \beta = 0$) [10], Li's and Kuhn's ($\beta = \gamma = -1/2, \alpha = 0$) [11] and the very recent Mustafa's and Mazharimousavi's ($\alpha = \gamma = -1/4, \beta = -1/2$) [3]. Nevertheless, in this work we shall deal with these orderings irrespective to their classifications of being 'good-' (i.e. satisfying the continuity requirements on the wavefunction, mentioned above, and surviving the Dutra's and Almeida's [7] reliability test) or 'to-be-discarded-' orderings (i.e. not satisfying the continuity requirements on the wavefunction and/or failing the Dutra's and Almeida's [7] reliability test). The reader is advised to refer to, e.g., Mustafa and Mazharimousavi [3] for more details.

The growing interest in the non-Hermitian pseudo-Hermitian Hamiltonians with real spectra [13–21], on the other hand, have inspired our recent work on PDM first-order-intertwining operator and η-weak-pseudo-Hermiticity generators [12]. A Hamiltonian H is pseudo-Hermitian if it obeys the similarity transformation $\eta H\eta^{-1} = H^{\dagger}$, where η is a Hermitian invertible linear operator and (†) denotes the adjoint. The existence of real eigenvalues is realized to be associated with a non-Hermitian Hamiltonian provided that it is an η-pseudo-Hermitian:

$$\eta H = H^{\dagger}\eta, \tag{4}$$

with respect to the nontrivial 'metric' operator $\eta = O^{\dagger}O$, for some linear invertible operator $O : \mathcal{H}\rightarrow\mathcal{H}$ ($\mathcal{H}$ is the Hilbert space). However, under some rather mild assumptions, we may even relax H to be an η-weak-pseudo-Hermitian by not restricting η to be Hermitian (cf, e.g., [17]), and linear and/or invertible (cf, e.g., [12, 18–20]).

Whilst in the non-Hermitian pseudo-Hermitian Hamiltonians neighborhood [13–23], the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians (i.e. a Bender's and Boettcher's [13] initiative on the so-called nowadays $\mathcal{PT}$-symmetric quantum mechanics) are unavoidably in point. They form a subclass of the non-Hermitian pseudo-Hermitian Hamiltonians (where $\mathcal{P}$ denotes parity and $\mathcal{T}$ mimics the time reversal). Namely, if $\mathcal{PT}H\mathcal{PT} = H$ and if $\mathcal{PT}\Phi(x) = \pm\Phi(x)$ the eigenvalues turn out to be real. However, if the latter condition is not satisfied the eigenvalues appear in complex-conjugate pairs (cf, e.g., [13]).

In this work, we consider (in section 2) a complexified von Roos Hamiltonian (1) (i.e. $\tilde{V}(x) \longrightarrow \tilde{V}(x) + \mathrm{i}W(x)$) regardless of the nature of the ordering of the ambiguity parameters as to being 'good' or 'to-be-discarded' ones. A Hermitian first-order differential PDM-intertwining operator is used to obtain the corresponding non-Hermitian η-weak-pseudo-Hermitian PDM-Hamiltonian. The related *reference-target* non-Hermitian η-weak-pseudo-Hermitian Hamiltonians' map is also given in the same section. Yet, in connection with the resulting effective *reference* potential, a *'user-friendly'* form is suggested (in section 3) to serve for exact-solvability of some non-Hermitian η-weak-pseudo-Hermitian PDM-Hamiltonians. Such a *user-friendly* form turns out to imply that there is always a set of isospectral *target* η-weak-pseudo-Hermitian PDM-Hamiltonians associated with *'one'* exactly solvable *reference*

η-weak-pseudo-Hermitian PDM-Hamiltonian. We use (in the same section) two illustrative examples (i.e. a complexified $\mathcal{PT}$-symmetric Scarf-II and a periodic-type $\mathcal{PT}$-symmetric Samsonov–Roy potentials) as *reference* models and report the corresponding sets of isospectral η-weak-pseudo-Hermitian *target* -Hamiltonians. Section 4 is devoted for the concluding remarks.

2. An η-intertwiner and η-weak-pseudo-Hermitian Hamiltonians' *reference-target* map

A complexification of the potential $\tilde{V}(x)$ in (1) may be achieved by the transformation $\tilde{V}(x) \longrightarrow \tilde{V}(x) + \mathrm{i}W(x)$, where $\tilde{V}(x), W(x) \in \mathbb{R}$ and $\mathbb{R} \ni x \in (-\infty, \infty)$. Hence, Hamiltonian (1) becomes non-Hermitian and reads

$$H = -\mu(x)^2\partial_x^2 - 2\mu(x)\mu'(x)\partial_x + \tilde{V}(x) + \mathrm{i}W(x), \tag{5}$$

with $\mu(x) = \pm 1/\sqrt{M(x)}$. A Hermitian first-order intertwining PDM-differential operator (cf, e.g., Mustafa and Mazharimousavi [12] on the detailed origin of this PDM-operator) of the form

$$\eta = -\mathrm{i}[\mu(x)\partial_x + \mu'(x)/2] + F(x); \qquad F(x), \mu(x) \in \mathbb{R} \tag{6}$$

would result, when used in (4),

$$W(x) = -\mu(x)F'(x), \tag{7}$$

$$\tilde{V}(x) = -F(x)^2 - \tfrac{1}{2}\mu(x)\mu''(x) - \tfrac{1}{4}\mu'(x)^2 + \alpha_\circ, \tag{8}$$

where $\alpha_\circ \in \mathbb{R}$ is an integration constant. One may then recast $V(x)$ as

$$V(x) = \alpha_\circ - F(x)^2 + \left(\tfrac{1}{2} + \beta\right)\mu(x)\mu''(x) + \left[4\alpha(\alpha+\beta+1) + \beta + \tfrac{3}{4}\right]\mu'(x)^2. \tag{9}$$

One should, nevertheless, be reminded that an anti-Hermitian first-order operator of the form $\eta = \mu(x)\partial_x + \mu'(x)/2 + \mathrm{i}F(x)$ will do exactly the same job (cf, e.g., Mustafa and Mazharimousavi [12]). Moreover, as a result of this intertwining process, a non-Hermitian η-weak-pseudo-Hermitian Hamiltonian is obtained.

We may now consider our non-Hermitian η-weak-pseudo-Hermitian Hamiltonian in (5), along with (7) and (8), in the one-dimensional Schrödinger equation

$$H_x\psi(x) = E\psi(x) \tag{10}$$

and construct the so-called *reference-target* η-weak-pseudo-Hermitian Hamiltonians' map (equation (10) is the so-called *target* Schrödinger equation). A task that would be achieved by the substitution

$$\psi(x) = \varphi(q(x))/\sqrt{\mu(x)}, \tag{11}$$

to imply, with the requirement

$$q'(x) = 1/\mu(x) \tag{12}$$

that removes the first-order derivative $\partial_q\varphi(q)$, a so-called *reference* Schrödinger equation

$$-\partial_q^2\varphi(q(x)) + [\tilde{V}_{\mathrm{eff}}(q(x)) - E]\varphi(q(x)) = 0, \tag{13}$$

where

$$\tilde{V}_{\mathrm{eff}}(q(x)) = (\beta+1)\mu(x)\mu''(x) + [4\alpha(\alpha+\beta+1)+\beta+1]\mu'(x)^2 - F(x)^2 + \alpha_\circ - \mathrm{i}\mu(x)F'(x). \tag{14}$$

It is evident that our η-weak-pseudo-Hermitian *reference*-Hamiltonian

$$H_q = -\partial_q^2 + \tilde{V}_{\text{eff}}(q), \tag{15}$$

of (13), shares exactly the same spectrum of the η-weak-pseudo-Hermitian *target*-Hamiltonian

$$H_x = -\mu(x)^2\partial_x^2 - 2\mu(x)\mu'(x)\partial_x - \tfrac{1}{4}\mu'(x)^2 - \tfrac{1}{2}\mu(x)\mu''(x) + \tilde{V}_{\text{eff}}(x), \tag{16}$$

defined in (5), (7) and (8), where

$$\tilde{V}_{\text{eff}}(x) = \alpha_\circ - F(x)^2 - \mathrm{i}\mu(x)F'(x),$$

and H_q and H_x are isospectral. Nevertheless, one should keep in mind that H_q and H_x may very well interchange their roles as to being a *reference* or a *target* Hamiltonians. That is, it might just happen that H_x is exactly solvable and in this case H_x becomes a *reference*-Hamiltonian and H_q plays the role of being a *target*-Hamiltonian.

3. PDM-functions admitting isospectrality

It is obvious that the effective *reference* potential in (14) suggests that the choice of

$$(\beta+1)\mu(x)\mu''(x) + [4\alpha(\alpha+\beta+1)+\beta+1]\mu'(x)^2 = 0, \tag{17}$$

would imply a '*user-friendly*' effective *reference* potential of the form

$$\tilde{V}_{\text{eff}}(q) = \alpha_\circ - F(q)^2 - \mathrm{i}F'(q). \tag{18}$$

Hence

$$\mu'(x)\mu(x)^\delta = \text{const.}$$

and

$$\mu(x) = [C_1x + C_2]^{1/(\delta+1)}; \qquad \delta = \left[4\alpha + 1 + \frac{4\alpha^2}{\beta+1}\right], \tag{19}$$

where C_1 and C_2 are two constants and $C_1, C_2 \in \mathbb{R}$. Nevertheless, one should note that the Ben Daniel's and Duke's ($\alpha = \gamma = 0, \beta = -1$) ordering (although $\beta = -1$ is not allowed by (19) but satisfies (17)) has already been discussed by Mustafa and Mazharimousavi [12]. Hence, the Ben Daniel's and Duke's ordering shall not be considered in the forthcoming studies. Moreover, under such mass settings, we may report that; for Gora's and Williams' ($\beta = \gamma = 0, \alpha = -1$) and Li's and Kuhn's ($\beta = \gamma = -1/2, \alpha = 0$) orderings $\delta_{\text{GW}} = \delta_{\text{LK}} = 1$, for Zhu's and Kroemer's ($\alpha = \gamma = -1/2, \beta = 0$) ordering $\delta_{\text{ZK}} = 0$, and for Mustafa's and Mazharimousavi's ($\alpha = \gamma = -1/4, \beta = -1/2$) ordering $\delta_{\text{MM}} = 1/2$.

Moreover, it is evident that the position-dependent-mass $M(x)$ under the current settings is strictly determined through (17) and consequently through (19) to read

$$M(x) = \mu(x)^{-2} = [C_1x + C_2]^{-2/(\delta+1)}. \tag{20}$$

Hence, one may safely conclude that this PDM form identifies a class of isospectral position-dependent-mass functions satisfying the effective *reference* potential $\tilde{V}_{\text{eff}}(q)$ of (18), for each form of the η-weak-pseudo-Hermiticity generator $F(q)$, and implies

$$q(x) = \int^x \mu(y)^{-1}\,\mathrm{d}y = \begin{cases} \dfrac{(\delta+1)}{\delta C_1}[C_1x + C_2]^{\delta/(\delta+1)}; & \text{for } \delta \neq 0 \\ \dfrac{1}{C_1}\ln(C_1x + C_2); & \text{for } \delta = 0 \end{cases}. \tag{21}$$

However, it should be noted that this case (i.e. $M(x)$ is strictly determined) is unlike the one we have very recently considered in [12], where Ben Daniel's and Duke's ordering (i.e.

$\alpha = \gamma = 0, \beta = -1$) was used and the position-dependent-mass was left arbitrary instead (but, of course, a positive-valued function). Yet, one should clearly observe that the form of our $\tilde{V}_{\text{eff}}(q)$ in (18) depends only on the choice of our η-weak-pseudo-Hermiticity generator $F(q)$. It is advised that such a choice should be oriented so that an exactly solvable η-weak-pseudo-Hermitian *reference* Hamiltonian is obtained. Consequently, a set of exactly solvable isospectral η-weak-pseudo-Hermitian *target*-Hamiltonians of (16) would result and depend only on the class of the strictly determined position-dependent-mass functions in (20). Two illustrative examples are in order.

3.1. A complexified $\mathcal{PT}$-symmetric Scarf-II model

Let us recollect that an η-weak-pseudo-Hermiticity generator (cf, e.g., Mustafa and Mazharimousavi [12]) of the form

$$F(q) = -V_2 \operatorname{sech} q \Longrightarrow F'(q) = V_2 \operatorname{sech} q \tanh q \tag{22}$$

would yield (with $\alpha_\circ = 0$) a *reference* effective complexified $\mathcal{PT}$-symmetric Scarf-II potential of the form

$$\tilde{V}_{\text{eff}}(q) = -V_2^2 \operatorname{sech}^2 q - \mathrm{i} V_2 \operatorname{sech} q \tanh q; \qquad \mathbb{R} \ni V_2 \neq 0. \tag{23}$$

Which, in turn, would imply a *target* effective potential of the form

$$\tilde{V}_{\text{eff}}(x) = -4V_2^2 \frac{f(x)^2}{(f(x)^2+1)^2} \mp 2\mathrm{i} V_2 \frac{f(x)(f(x)^2-1)}{(f(x)^2+1)^2}, \tag{24}$$

where $f(x) = \pm \exp[q(x)]$, with $q(x)$ given in (21). In this case, the *target* effective potentials in (24) form a set of isospectral η-weak-pseudo-Hermitian Hamiltonians

$$\begin{aligned} H_x = &-\mu(x)^2 \partial_x^2 - 2\mu(x)\mu'(x)\partial_x - \frac{1}{4}\mu'(x)^2 - \frac{1}{2}\mu(x)\mu''(x) \\ &- 4V_2^2 \frac{f(x)^2}{(f(x)^2+1)^2} \mp 2\mathrm{i} V_2 \frac{f(x)(f(x)^2-1)}{(f(x)^2+1)^2}. \end{aligned} \tag{25}$$

All of which share (with $\mu(x)$ as defined in (19)) the same eigenvalues readily reported in [12, 17] as

$$E_n = -\left[|V_2| - n - \frac{1}{2}\right]^2; \qquad n = 0, 1, 2, \ldots, n_{\max} < (|V_2| - 1/2). \tag{26}$$

3.2. A periodic-type $\mathcal{PT}$-symmetric Samsonov–Roy model

We may also recycle our η-weak-pseudo-Hermiticity generator

$$F(q) = -\frac{4}{3\cos^2 q - 4} - \frac{5}{4}, \tag{27}$$

that implies (with $\alpha_\circ = 0$) an effective periodic-type $\mathcal{PT}$-symmetric Samsonov's and Roy's [12, 14] *reference* potential

$$\tilde{V}_{\text{eff}}(q) = -\frac{6}{[\cos q + 2\mathrm{i} \sin q]^2} - \frac{25}{16}; \qquad \mathbb{R} \ni q \in (-\pi, \pi). \tag{28}$$

This results, in effect, a *target* effective potential of the form

$$\tilde{V}_{\text{eff}}(x) = -\frac{6}{[g(x) - 2\mathrm{i}\mu(x)g'(x)]^2} - \frac{25}{16}, \tag{29}$$

where $g(x) = \cos(q(x))$, $\mu(x)$ and $q(x)$ are as given in (19) and (21), respectively. Hence, the set of η-weak-pseudo-Hermitian *target* Hamiltonians

$$H_x = -\mu(x)^2\partial_x^2 - 2\mu(x)\mu'(x)\partial_x - \frac{1}{4}\mu'(x)^2 - \frac{1}{2}\mu(x)\mu''(x) \\ - \frac{6}{[g(x) - 2\mathrm{i}\mu(x)g'(x)]^2} - \frac{25}{16} \tag{30}$$

are isospectral and share the eigenvalues [12, 14]

$$E_n = \frac{n^2}{4} - \frac{25}{16}; \qquad n = 1, 3, 4, 5, \ldots, \tag{31}$$

with a missing $n = 2$ state (the details of which can be found in Samsonov and Roy [14]).

4. Concluding remarks

As long as η-weak-pseudo-Hermitian Hamiltonians are in point, their solvability-nature/type (i.e., e.g., exact-, quasi-exact-, conditionally exact, etc) is still fresh and not yet adequately explored. Amongst is the η-weak-pseudo-Hermitian von Roos PDM-Hamiltonian. In this work, we tried to (at least) partially fill this gap and add a flavor into such solvability territories of the η-weak-pseudo-Hermitian Hamiltonians associated with position-dependent-mass settings.

In addition to mapping our η-weak-pseudo-Hermitian *target*-Hamiltonians H_x into η-weak-pseudo-Hermitian *reference*-Hamiltonians H_q (that share the same spectra for H_x and is advised to be exactly solvable), we have suggested a *'user-friendly'* form (in $\tilde{V}_{\text{eff}}(q)$ of (18)) for the *reference-target* η-weak-pseudo-Hermitian PDM-Hamiltonians' map. The usage of which is exemplified through a non-Hermitian $\mathcal{PT}$-symmetric Scarf II and a non-Hermitian $\mathcal{PT}$-symmetric Samsonov–Roy periodic-type models. It is observed that for each of these models there is a set of exactly solvable isospectral *target* η-weak-pseudo-Hermitian PDM-Hamiltonians (documented in (25) for Scarf II and in (30) for Samsonov–Roy). Hereby, it should be noted that the isospectrality among the η-weak-pseudo-Hermitian Hamiltonians in (16) is only manifested by the PDM choice of (17).

Of course there are other choices that might lead to some 'user friendly' forms of the effective potential in (14). The feasibility of the associated isospectrality should always be explored, therefore. For example, the choice

$$F(x) = \mu'(x) \Longrightarrow \mu(x) = \int^x F(y)\,\mathrm{d}y, \tag{32}$$

would imply an effective potential of the form

$$\tilde{V}_{\text{eff}}(q) = -\mathrm{i}F'(q) + (\beta + 1)F'(q) + [4\alpha(\alpha + \beta + 1) + \beta]F(q)^2 + \alpha_\circ, \tag{33}$$

Apart from the ambiguity parameters' setting of $\beta = -1$ (and consequently $\alpha = \gamma = 0$ by the von Roos constraint $\alpha + \beta + \gamma = -1$) considered by Mustafa and Mazharimousavi in [12], we were unlucky to find any illustrative example that can be classified as 'successful' for such an effective potential form (33). Nonetheless, the corresponding *target* isospectral set of η-weak-pseudo-Hermitian PDM-Hamiltonians is anticipated to be feasibly large (as documented by (32)) and not restricted to the position-dependent-mass form (unlike the case of $\tilde{V}_{\text{eff}}(q)$ of (18), which is restricted to the position-dependent-mass function $M(x)$ in (20)).

Moreover, we may report that a generating function $F(q) = a\exp(-q)$ would lead to (with $\alpha_\circ = 0$) to

$$\tilde{V}_{\text{eff}}(q) = -a^2\exp(-2q) + \mathrm{i}a\exp(-q) \tag{34}$$

of (18), and

$$\tilde{V}_{\text{eff}}(q) = a^2[4\alpha(\alpha+\beta+1)+\beta]\exp(-2q) - a(\beta+1-i)\exp(-q) \tag{35}$$

of (33). The bound-states of the former in (34) (a non-Hermitian Morse model) are reported to form an empty set of eigenvalues and, hence, labeled as 'unfortunate' for it leads to an empty set of isospectral η-weak-pseudo-Hermitian *target*-Hamiltonians (cf, e.g., [12, 22, 23]). The latter in (35), on the other hand, does not fit into any of the 'so-far-known' exactly solvable non-Hermitian Morse-type models, to the best of our knowledge. These two models form open problems (if their bound-state solutions exist at all), therefore.

Finally, one may add that the current strictly determined set of *target* effective potentials $\tilde{V}_{\text{eff}}(x)$ in (24) forms a subset of the *target* effective potentials reported in equations (25) and (26) by Mustafa and Mazharimousavi [12]. A similar trend is also observed for $\tilde{V}_{\text{eff}}(x)$ in (29) as it forms a subset of the effective potentials in equations (34) and (35) of [12]. Hence, the scenario of the *energy-levels crossing* and the feasible manifestation of the *flown away states* discussed in [12] remains effective, as far as our two illustrative examples are concerned.

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